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SPIN AND RELATIVITY: A SEMICLASSICAL MODEL FOR ELECTRON SPIN

ESPÍN Y RELATIVIDAD: UN MODELO SEMICLÁSICO PARA EL ESPÍN DEL ELECTRÓN

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RESUMEN

La relación cuántica $m_0c^2 = \hbar|\omega_0|$ puede ser considerada como la equivalencia entre dos expresiones para la energía en reposo de la partícula, si ω_0 se considera la velocidad angular de giro de partículas en su marco en reposo. La invariancia del intervalo relativista espacio- tiempo $ds^2 = c^2dt^2 - dr^2$ para tal movimiento de espín (isotropía espacial) conduce al impulso de espín $S_z = \hbar / 2$ para todas las partículas sin estructura, independientemente de sus valores de masa. La inercia es una propiedad intrínseca debido al movimiento de spin de las partículas. Los signos de los valores de masa que se producen en las soluciones de la ecuación de Dirac podrían estar relacionados con la orientación del espín, según lo sugerido por la relación fundamental $\pm m_0c^2 = \pm\hbar|\omega_0|$. Además se refiere al electrón, y más concretamente con dos de las principales propiedades: su función de onda compleja, y su giro intrínseco. En su interpretación estándar no hay una clara imagen del espacio real de lo que es oscilante en la onda, o lo que está girando en el espín. De hecho, es la creencia generalizada de que ningún modelo sencillo puede dar cuenta de la rotación de espín de los electrones. Por el contrario, en el presente trabajo se muestra que un crudo modelo mecánico de rotación de vórtices coherentes explica cuantitativamente no sólo el espín, sino también la propia función de onda. Las consecuencias de esto son examinadas en este trabajo.

Palabras clave: Espín, relatividad, ecuación de Dirac, función de onda, modelo semiclásico.

ABSTRACT

The quantum relationship $m_0c^2 = \hbar|\omega_0|$ may be regarded as the equivalence between two expressions for the rest energy of the particle, if ω_0 is considered as the spin angular velocity of the particle in its rest frame. The invariance of the relativistic space-time interval $ds^2 = c^2dt^2 - dr^2$ to such a spin motion (space isotropy) leads to the spin momentum $S_z = \hbar / 2$ for all structureless particles irrespective of their mass values. The inertia is an intrinsic property due to the spin motion of the particles. The signs of the mass values occurring in the solutions of the Dirac equation might be related to the orientation of the spin motion, as suggested by the fundamental relationship $\pm m_0c^2 = \pm\hbar|\omega_0|$. In addition, it deals with the electron, and more specifically with two key properties: its complex wavefunction and its intrinsic spin. In the standard interpretation, there is no clear real-space picture of what is oscillating in the wave, or what is rotating in the spin. Indeed, it is generally believed that no simple model of rotation can account for the spin of the electron. On the contrary, the present paper shows that a crude mechanical model of coherently rotating vortices can account quantitatively not only for spin, but also for the wavefunction itself. The implications of this are discussed in this paper.

Keywords: Spin, relativity, Dirac equation, wavefunction, semiclassical model.

SPIN AND RELATIVITY

As we know, the spin cannot be motivated in the frame of classical mechanics. Even in the nonrelativistic quantum theory, the nature of spin remains unclear. The spin results solely from Dirac's equation [1]. Although the Pauli and Dirac matrices undoubtedly show the spin existence, there is some mystery as to the physical origins of and

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in the visualization of the spin [2, 3]. One may says that spin is an intrinsic property of the matter. It must have to do with relativity even though this connection is not entirely understood [4].

In this paper, I shall try to sketch a simple motivation for the existence of spin starting from the fundamental relationship, and namely:

$$m_0 c^2 = \hbar |\omega_0| \tag{1}$$

where m_0 is the rest mass and m_0c^2 the rest energy of the particle.

If the particle is considered as being a physical torus spinning with angular velocity ω_0 (Figure 1a, and 1b), the right-hand side of equation (1) should be regarded as another relativistic expression for the rest energy.

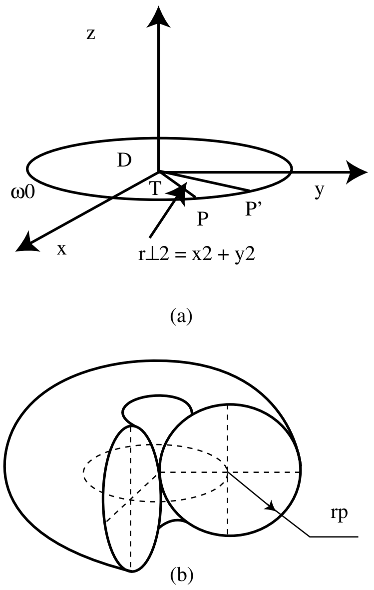

Figure 1. a) A spinning reference frame. b) A torus electron model

We may argue the existence of the spin motion in the space-time frame of the particle as follows: unlike space coordinates, time is not directly measurable (observable). The simplest way to estimate time is to consider uniform motion. One can obtain time by comparing covered distances ([5]: time is the number of motion). This leads to the necessity of introducing motion in the space-time reference frame. The only allowed motion in the rest frame of the particle should be that of rotation (spin).

Therefore, it is reasonable to assume that the rest energy of a particle is related to its spin motion, which is only allowed in that system [3]. This reasoning allows us to regard $\hbar|\omega_0|$ as an equivalent expression for the rest energy of the particle.

The particle as a moving object must also obey another fundamental relationship, namely the relativistic elementary "space-time interval" between the physical events of the particle:

$$ds^2 = c^2 dt^2 - dr^2 \tag{2}$$

Every physical process, such as translation, rotation, etc., must be related to expression (2). The invariance of ds^2 to uniform translation (space homogeneity) leads to the Lorentz corrections [6].

Let us now consider the uniform rotation (spin) of the reference frame, with angular velocity ω_0 in x,y-plane around the z-axis, as shown in Figure 1. In this case, we have

$$x \to x'; y \to y'; z = z';$$
$$t \to t' = t + dt; \omega_0 = \frac{d\varphi}{dt} = const;$$

and ds^2 (equation 2) becomes

$$ds'^2 = (c^2 - r_\perp^2 |\omega_0|^2) - 2\omega_0 (ydx - xdy)dt - dr^2 \tag{3}$$

where $r_\perp^2 = x^2 + y^2$ represents the radius perpendicular to the rotation z-axis, i.e. the distance from origin to the points P, P', etc. (Figure 1).

Expression (3) can easily be derived.[7] Note that the linear velocity $u = r_\perp |\omega_0|$ must obey the restriction imposed by the special relativity, i.e. $c^2 > r_\perp^2 |\omega_0|^2$. For the limit case $c = r_\perp |\omega_0|$, the rotating space becomes 'closed' with the lateral radius $r_\perp$. This is all what Fock mentioned [7].

But such a rotating empty space is physically meaningless. We must therefore actually ascribe this rotation to the particle situated in the origin of this space. For that particle, the set of points P, P', etc. for which $c = r_\perp |\omega_0|$, should be considered the closure (frontier) of the particle. We do not know too much about the shape of a spinning particle, considered as being structureless, but we can at least define for it a radial extension equal to:

$$r_{\perp} = c / |\omega_0| \quad (4)$$

From equations (1) and (4), we have:

$$r_{\perp} = \hbar / m_0 c = \lambda\!\!\!^{-}_c \text{ (Compton radius).} \quad (5)$$

This result shows that all structureless particles with rest mass cannot be pointlike. For the limit case, $c = r_{\perp} |\omega_0|$, expression (3) becomes:

$$ds'^2 = 2\omega_0 (xdy/dt - ydx/dt)dt^2 - dr^2 \quad (6)$$

We have already mentioned the invariance of to the uniform translation (space homogeneity). Let us now consider the invariance of ds^2 to the uniform rotation (space isotropy). In other words, for a noninteracting spinning particle, space must remain unaffected by the uniform rotation. From the invariance condition ds'^2 (equation 6) ≡ ds^2 (equation 2), we obtain:

$$2|\omega_0|(xdy/dt - ydx/dt) = c^2 . \quad (7)$$

If we use now the fundamental relationship from equation (1), we have

$$S_z = m_0 (xdy/dt - ydx/dt) = m_0 |r \times u|_z = \hbar / 2 , \quad (8)$$

where S_z is the classical expression for the z-component of angular momentum. The result is interesting. This mainly shows that the $\hbar/2$ value of the angular (spin) momentum preserves the space isotropy. It must be universal and characteristic for all structureless particles with finite rest mass, independent of their mass values.

If the two possible rotations around z-axis are considered ($\omega_0 \in \{+|\omega_0|; -|\omega_0|\}$) corresponding to $x \to y$ and $y \to x$ rotations, both the $\pm\hbar/2$ values conserve the space isotropy. The time reversal t→-t in (8) leads to $-\hbar/2$ value. Note that for $\pm|\omega_0|$ values equation (1) becomes $\pm\hbar|\omega_0| = \pm m_o c^2$. The mass values $\pm m_0$ occurring in Dirac's equation might actually be related to the rotation sense of the spin angular velocity $|\omega_0|$.

Moreover, according to (1) the rest mass m_0 is tightly connected with the spin motion represented by ω_0. Therefore, a structureless elementary particle with a finite rest mass and radial extension behaves as a small mechanical top, its inertial properties not necessarily being conditioned by the gravitational interaction with the matter in universe (Mach's principle).

A SEMICLASSICAL MODEL FOR ELECTRON SPIN

First, consider an electron with its center of mass at rest, but spinning. The simplest possible model is a spinning solid torus (figure 1b). Based on the goal of having this describe the electron wavefunction, one expects that the angular velocity is given by the Planck-Einstein relation $E = \hbar\omega$. Since this is a real physical rotation, the zero of energy is not arbitrary as in standard nonrelativistic quantum mechanics, but must be given by the relativistic rest energy $E = mc^2$. (This also has the property of being relativistically covariant when we transform later to a moving reference frame.) For rotation of a solid torus of radius R, the linear velocity on the equator is $u = R\omega = Rmc^2/\hbar$. But clearly, u can be no greater than the speed of light c. This is a natural cutoff, and provides an estimate of the maximum size of this spinning ball:

$$R_{\max} = c/\omega = \hbar / mc = R_c \quad (9)$$

This is the Compton wavelength R_c of the electron ˜0.4pm, which is much smaller than the typical Å scale that characterizes atomic orbitals (1Å=100pm). If we want to model an extended electron state, then clearly R_c is too small.

Consider instead an extended state consisting of a parallel array of torus vortices (see figure 1), each a solid body of radius R_c rotating around its axis at $\omega = mc^2/\hbar$. For simplicity here, assume that there are N identical vortices, each of mass *mv = m/N*. The angular momentum of each vortex is then given by

$$L_v = I\omega = \frac{1}{2} m_v R_c^{\,2} \omega = \hbar / 2N , \quad (10)$$

where we have taken the moment of inertia $I = \frac{1}{2}mR^2$ for a cylinder of uniform mass density. This is a crude semi-relativistic model, but it does in fact give the proper value for the total angular momentum for the electron, $S = \hbar / 2$.

One can also estimate the magnetic moment of the electron from this model. Treating the rotating charge per vortex $q_v = e / N$ as a current $i_v = q_v \omega / 2\pi$, one obtains simply

$$\mu = N i_v A_v = \left(e\omega / 2\pi\right)\left(\pi R_c{}^2\right) = e\hbar / 2m = \mu_B \quad (11)$$

where μ_B is the Bohr magneton and A_v is the cylindrical cross sectional area per vortex. Again, this is the correct result, perhaps fortuitously, but it does suggest that this crude model may incorporate much of the essential physics.

These calculations require only that all of the torus are rotating at the same frequency around parallel axes. But in addition, it is reasonable to assume a coherent state where all of them are rotating in-phase as well, as suggested in figure 1. This requires a rotating vector field *A(r,t)*. Furthermore, it is not necessary to assume that the vortices have identical masses. More generally, one could have a density function $\rho(r)$, which would go as the square of the field amplitude $A(r)$, analogously to the energy density in electromagnetic waves.

Now the phase angle $\theta(t) = Et / \hbar$ is constant across the entire electron, but that can also be relaxed. Consider what happens when we Lorentz-transform to a reference frame moving with velocity v. Locations that are in phase in the rest frame will not in general be in phase in the moving frame. The proper way to deal with this is to make the phase angle relativistically invariant, so that

$$Et \Rightarrow E't' - p' \cdot r' \quad (12)$$

where in the usual way $E' = \gamma mc^2 \approx mc^2 + \frac{1}{2}m\mathrm{v}^2$, $p' = \gamma m\mathrm{v} \approx m\mathrm{v}$ is the momentum, $\gamma = \left(1 - \mathrm{v}^2 / c^2\right)^{-1/2}$, and the approximate forms are for $\mathrm{v} \ll c$. This is invariant because $(E/c, p)$ and (ct, r) are relativistic 4-vectors, and the phase angle goes as their inner product. So now the rotating phase angle takes the form

$$\theta(r,t) = (Et - p \cdot r) / \hbar \quad (13)$$

This corresponds to a plane wave with wavelength $\lambda = h/p$, which is well known as the de Broglie wavelength. Note that this follows directly from the earlier assumption that the rotation frequency is given by $mc^2 / \hbar$.

Once we have a wave satisfying the Einstein-deBroglie relations, the rest of quantum mechanics follows naturally. We have a rotating vector field given by a spin axis (assumed to be uniform), an amplitude $A(r,t)$, and a rotating phase angle $\theta(r,t)$. If we compare to the standard complex wavefunction in quantum mechanics, $\Psi(r,t) = |\Psi| \exp(i\varphi)$, and map A and θ onto $|\Psi|$ and φ, we have a rotating wavefunction which satisfies the time-dependent Schrödinger equation.

For example, consider a rotating vector field of the form

$$A(r,t) = A_0\left[u_x \cos(kz - \omega t) \pm u_y \sin(kz - \omega t)\right], \quad (14)$$

(u_x and u_y are the unit vectors in the x- and y-directions), which represents a plane wave traveling in the z-direction with spin also in the z-direction (figure 2). This is a circularly polarized transverse wave, with either positive or negative helicity depending on whether the plus or minus sign is chosen. For fixed t, the tip of the vector follows a helix; for fixed z, circular rotation at an angular frequency ω of a vector of length A_0.

Now define $\theta = \arctan\left(A_y / A_x\right) = kz - \omega t$, and

$$\Psi(r,t) = A\exp(i\theta) = A\exp\left[i(kz - \omega t)\right], \quad (15)$$

and substitute this into the time-dependent Schrödinger equation with the rest-energy explicitly added:

$$i\hbar\partial\Psi / \partial t = H\Psi = \left(-\hbar^2 / 2m\right)\nabla^2\Psi + \left[mc^2 + V(r)\right]\Psi \quad (16)$$

The result is the simple, correct relation (for $\mathrm{v} \ll c$) that $\hbar\omega = \hbar^2 k^2 / 2m + mc^2$. Note also that the complex conjugate of Ψ might seem to yield negative energy, but really just represents the spin of the opposite sign.

Thus far the model has been limited to a single plane wave, but electrons are generally present in bound states, with standing waves instead of travelling waves. Consider for simplicity the one-dimensional particle-in-a-box, with the electron confined between $z = 0$ and $z = L$. The solution takes the form of discrete bound states given by the complex wavefunctions Ψ_n and equivalent vector fields A_n:

$$\Psi_n = \sin(n\pi z / L)\exp(-i\omega t) \tag{17}$$

$$A_n = \sin\left(n\pi z / L\right)\left(u_x \cos\omega t \pm u_y \sin\omega t\right) \tag{18}$$

Here n=1 corresponds to the ground state and n=2, 3,... to the excited states, and the quantized energies E_n are given as usual (but with the mc^2 offset) by

$$E_n = \hbar\omega_n = mc^2 + \hbar^2 k^2 / 2m = mc^2 + \hbar^2 (n\pi / L)^2 / 2m \tag{19}$$

and as before the ± corresponds to the two spin states. Note that this vector wavefunction has separated into two factors, the usual standing-wave envelope and the rotating phase vector. The negative values of the sine (for n>1) correspond to 180° shifts of the rotating phase.

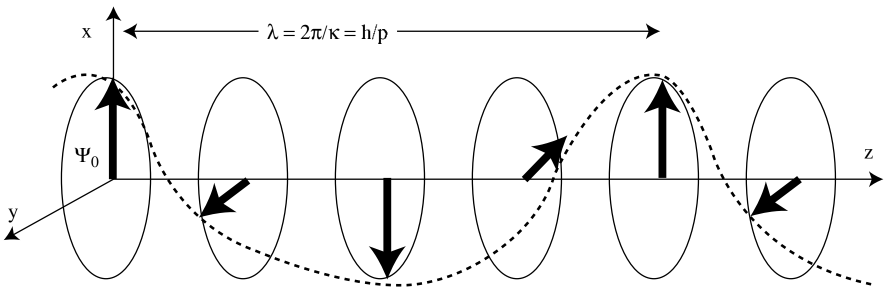

Figure 2. Picture of real-space helical wave representing electron with spin. Evolution of helix for wave propagating in z-direction.

DISCUSSION AND CONCLUSIONS

The wave example given above is based on a helical transverse wave, which is similar in form to a transverse electromagnetic wave which is circularly polarized like a chiral wave. Indeed, such a helical TEM wave carries angular momentum, and forms the classical limit of a photon [9, 10], with spin $\pm\hbar$ pointing along the direction of motion. However, unlike the case of the photon, one can transform to the rest frame of the electron, and from there to any other direction. In general, the electron spin axis would not be parallel to the momentum, and the rotating spin field vector would follow a general cycloidal motion rather than a simple helix. The spin and translational motions are essentially decoupled in this model (no spin-orbit interaction).

This model of coherently rotating vortices appears to account for the complex wavefunction of the electron [6]. This suggests that the spin picture may be substantially more general than simply a single electron, and that spin is fundamental to all of quantum mechanics. In that regard, it may not be a coincidence that all fundamental quantum particles seem to have spin. Certain mesons have spin-0, but they can be regarded as composites of spin-½ quarks. And certainly atoms with spin-0 show quantum effects. It is likely that the spins of the constituent components contribute their angular phase references to the composite system, even if the total spin cancels out.

One may speculate as to the physical basis for such a coherent vortex model. It seems to correspond to a very rigid state of an intrinsically rotating fluid. Such a rigid state may indicate a very strong cohesive energy associated with long-range phase coherence among the vortices. Since the lowest excitation of an electron involves creation of an electron-positron pair, this cohesive energy might be expected to be ~1MeV, larger than the rest energy of the electron itself.

Speculating even further, the existence of such a highly rigid state would have important implications for quantum measurement. Any local interaction that would alter the energy of part of an electron wavefunction would jeopardize this cohesive energy. This, in turn, would create an instability leading either to the rest of the electron being pulled into the interaction region, or alternatively to the expulsion of the electron from this region. This suggests a real dynamical process which may provide a physical basis for the "collapse of the wavefunction" in quantum measurement.

Finally, if this rotating spin field is mathematically equivalent to the usual Schrödinger equation, is it really just a matter of preference which representation we choose? Not entirely, because a real physical rotation, with a definite frequency and spatial fine structure, should be measurable. If one probes the behavior of electrons at frequencies ~10^{20} Hz = mc^2/h, particularly with a circularly polarized probe, one should expect to see a sharp resonance in some sort of spectral response, perhaps associated with spin-flip of the electron in a large magnetic field. Furthermore, the fine structure of the spin model identified a periodicity on the scale of $2R_c = 2\hbar / mc$, which would correspond to a momentum transfer $\hbar k = \pi mc \sim 1.5\,\mathrm{MeV} / c$. It would be interesting to see whether relevant measurements are consistent with the model described in this paper.

It is somewhat surprising that a simple mechanical model for spin was not presented in the early days of quantum mechanics. It seems that early researchers were discouraged by apparent rotation velocities greater than c [8]. It may be that the distributed coherent vortex model provides a way around these difficulties. More recently, Ohanian [9] showed that the relativistic Dirac equation is consistent with a distributed circular energy flow on a scale larger than R_c, which provides the basis for the electron spin and magnetic moment. The present semiclassical model is certainly cruder than the Dirac equation, but also reproduces these results within a more intuitive physical picture.

In conclusion, a new semiclassical picture for electron spin is presented, in which a spinning vector field, rotating at mc^2/h, is organized into a coherent array of rigidly rotating vortices on the scale of $R_c = \hbar / mc$. The vector field F maps onto the quantum wavefunction Ψ, providing for a unification of spin and quantum mechanics. It is further suggested that the coherent nature of this spin field may be associated with a cohesive energy, which in turn may play a key role in quantum measurement. While the specific details of this model remain crude, its clear intuitive physical picture may help to stimulate further research along similar lines. By dealing with specific real-space models, it may be possible to remove much of the abstraction and mystery from quantum theory.

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